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Gorenstein Flat Covers of Modules over Gorenstein Rings

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A left and right Noetherian ring R is called Gorenstein if both ${}_RR$ and R_R have finite injective dimensions. These rings were studied by Bass for the commutative case and Iwanaga for the noncommutative case. In this paper, we define Gorenstein flat modules over a Gorenstein ring. These modules are a generalization of flat modules. After discussing the properties of modules over a Gorenstein ring, we prove that every module over a Gorenstein ring has a Gorenstein flat cover or a minimal right approximation by a Gorenstein flat module in Auslander and Reiten's terminology. © 1996 Academic Press, Inc.

INTRODUCTION

For an associative ring R and a class of left R-modules $\mathscr{X}$, by Auslander and Buchweitz [2], Auslander and Reiten [3], and Enochs [9], we can define a right (left) $\mathscr{X}$-approximation or $\mathscr{X}$-precover (preenvelope) for an R-module. Therefore, we have the question of the existence of $\mathscr{X}$-precover (preenvelope) and $\mathscr{X}$-cover (envelope). The interest of this question becomes clear when the class $\mathscr{X}$ is specified. For instance, let $\mathscr{X} = \mathscr{E}$ be the class of all left injective R-modules. Then it is well known that $\mathscr{E}$-envelopes exist. These are just the injective envelopes (hulls) of modules in Eckmann–Schopf's sense [7]. Dually, if $\mathscr{X}$ is taken to be the class of all projective left R-modules, denoted $\mathscr{P}$, the notion of $\mathscr{P}$-covers agrees with that of projective covers (see Bass [4]). The existence problem for projective covers was completely solved by Bass in [4], and this lead to the extensive study on left perfect rings. Also, Enochs proved in [9] that $\mathscr{E}$-covers exist for all left R-modules if and only if R is a left Noetherian ring. Note that a ring R is left perfect if and only if all flat left R-modules are projective. Hence if $\mathscr{F}$ stands for the class of all flat left R-modules, it raised the question (still open) of the existence of flat covers (also see [9]).

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Flat covers are known to exist over commutative Noetherian rings of finite Krull dimension (see Xu [23]) and over right coherent rings of finite weak global dimension (see Enochs *et al.* [17]). In [2], Auslander and Buchweitz proved the existence of maximal Cohen−Macaulay approximations for finitely generated modules over a Cohen−Macaulay ring admitting a dualizing module. In [3], Auslander and Reiten worked out the existence of right $\mathscr{P}^{\infty}(\Lambda)$-approximations over an Artinian algebra Λ. Here $\mathscr{P}^{\infty}(\Lambda)$ is the class of all finitely generated left Λ-modules of finite projective dimension over Λ. Along these lines, by changing the choice of the class $\mathscr{X}$ the various existence problems of covers and envelopes were investigated by many authors from different point of views (also see [20, 14, 15, 5, and 22]). We see that the study of general covers and envelopes will enrich the homological methods in the theory of rings and modules, provide more invariants and unify all the well known existing envelopes and covers. In particular, when we restrict our concern to commutative Noetherian rings, the investigation of covers and envelopes will be of interest, and they will produce useful new invariants which are similar to Bass numbers and Betti numbers (see Robert [18], Ding [5, 6], and Xu [22]). We also see that the existence of covers (or envelopes) deeply depends on the structures of the chosen class and the ring itself.

In this paper, we focus on the Gorenstein flat covers of modules over a Gorenstein ring. A left and right Noetherian ring R is said to be Gorenstein if both ${}_RR$ and R_R have finite injective dimensions (see [4, 12, 11]). The Gorenstein rings were originally studied by Bass in [4] in the commutative case. A left R-module F is called Gorenstein flat if $\mathrm{Tor}_R^1(X, F) = 0$ for any right R-module X of finite flat dimension. This is a generalization of the notion of a flat module. Over a commutative Noetherian and Gorenstein ring, it was noted that Gorenstein flat modules are just those modules F having the Bass numbers $\mu_i(p, F) = 0$ whenever $ht(p) > i$, where $ht(p)$ means the height of prime ideals p of R (see the strongly cotorsion free modules in [22]). Gorenstein flat modules were first defined in [8] by complexes and idea which was motivated by Auslander's work in [1].

In Section 1, we provide the necessary notation and preliminaries. Then, in Section 2, we first discuss Gorenstein projective modules. These were originally called modules of zero G-dimension by Auslander in [1]. We then investigate the existence of Gorenstein projective precovers. It is worth noting that the existence of a Gorenstein projective precover does not always guarantee the existence of a Gorenstein projective cover. Finally, in Section 3, we prove that every module over a Gorenstein ring has a Gorenstein flat cover. In order to do this, we have to use a special Gorenstein projective precover of a module M (which is assured in Section

2) and then combine this result with a special diagram construction to get our result.

Throughout this paper, R denotes a Gorenstein ring. If R is Gorenstein and $n \geq 0$ is an upper bound for the injective dimensions of ${}_RR$ and R_R, R is said to be n-Gorenstein. It is known that if R is n-Gorenstein, so is $R[G]$ for any finite group G (see Iwanaga [12] and Eilenberg and Nakayama [19]). As usual, for an R-module M we use $\text{proj.dim}_R(M)$ for the projective dimension, $\text{inj.dim}_R(M)$ for the injective dimension, and $\text{f.dim}_R(M)$ for the flat dimension.

1. PRELIMINARIES

PROPOSITION 1.1 (Iwanaga [12]). *If R is n-Gorenstein and M is a left R-module, then the following are equivalent*:

(a) $\text{proj.dim}_R(M) < \infty$;

(b) $\text{proj.dim}_R(M) \leq n$;

(c) $\text{inj.dim}_R(M) < \infty$;

(d) $\text{inj.dim}_R(M) \leq n$;

(e) $\text{f.dim}_R(M) < \infty$;

(f) $\text{f.dim}_R(M) \leq n$.

Remark 1.2. Iwanaga stated the equivalence (a)–(d) in [9]. However, it is not difficult to see that (e) and (f) can be added to the list.

By the above, we know that for an n-Gorenstein ring R, the class of all left R-modules which have finite projective dimension, the class of all left R-modules which have finite injective dimension and the class of all left R-modules which have finite flat dimension are all the same class. We use $\mathscr{C}$ to denote this class. If X is in $\mathscr{C}$, we say X has all these dimensions finite. It is easy to see that the class $\mathscr{C}$ is closed under extensions, direct sums and summands, direct products and direct limits.

Let $\mathscr{X}$ be a class of left R-modules which is closed under extensions, direct summands and isomorphisms. For a left R-module M, we recall the definition of an $\mathscr{X}$-precover and $\mathscr{X}$-cover of M.

DEFINITION 1.1. Let $\varphi\colon X \to M$ be a linear map with $X \in \mathscr{X}$. The pair (φ, X) is called an $\mathscr{X}$-cover of M if the following conditions hold:

(1) $\text{Hom}_R(X', X) \to \text{Hom}_R(X', M) \to 0$ is exact for any X' in $\mathscr{X}$. In other words, any linear map $X' \to M$ with X' in $\mathscr{X}$ can be lifted to a linear map $X' \to X$.

(2) For any endomorphism f of X with $\varphi = \varphi f$, f must be an automorphism of X.

The pair (φ, X) is called an $\mathscr{X}$-precover of M if (1) holds (and perhaps not (2)).

For convenience we may simply call X or φ an $\mathscr{X}$-(pre)cover of M if there is no confusion. It is easy to check that if an $\mathscr{X}$-cover exists, it is unique up to isomorphism. Dually, we have the notion of $\mathscr{X}$-envelopes and $\mathscr{X}$-preenvelopes. For completeness, we state the following definition.

DEFINITION 1.2. Let $\varphi\colon M \to X$ be a linear map with $X \in \mathscr{X}$. The pair (φ, X) is called an $\mathscr{X}$-envelope of M if the following are satisfied:

(1) $\operatorname{Hom}(X, X') \to \operatorname{Hom}(M, X') \to 0$ is exact for any $X' \in \mathscr{X}$. In other words, any linear map $M \to X'$ with $X' \in \mathscr{X}$ can be extended to a linear map $X \to X'$;

(2) For any endomorphism f of X with $\varphi = f\varphi$, f must be an automorphism of X.

The pair (φ, X) is called an $\mathscr{X}$-preenvelope of M if (1) holds (and perhaps not (2)).

We have the same notes as we did above for $\mathscr{X}$-covers (precovers). Let ${}_R\mathscr{M}$ be the category of all left R-modules. For a class $\mathscr{L}$ of left R-modules which are closed under extensions, direct summands and isomorphisms, we define its right and left orthogonal classes as

$$\mathscr{L}^{\perp} = \left\{Y \in {}_R\mathscr{M} | \operatorname{Ext}^1_R(X, Y) = 0,\ X \in \mathscr{L}\right\}$$

$$^{\perp}\mathscr{L} = \left\{Y \in {}_R\mathscr{M} | \operatorname{Ext}^1_R(Y, X) = 0,\ X \in \mathscr{L}\right\}$$

The lemma below is useful for our purpose.

PROPOSITION 1.3 (Wakamatsu's Lemma [3]). *M is in ${}_R\mathscr{M}$, $X \to M$ is an $\mathscr{X}$-cover of M, then $K = \ker(X \to M) \in \mathscr{X}^{\perp}$.*

Proof. See Auslander and Reiten [3]. ∎

DEFINITION 1.3. R is n-Gorenstein. A left R-module G is called Gorenstein injective if $G \in \mathscr{C}^{\perp}$; that is, $\operatorname{Ext}^1_R(X, G) = 0$ for any $X \in \mathscr{C}$. A left R-module P is called Gorenstein projective if $P \in {}^{\perp}\mathscr{C}$; that is, $\operatorname{Ext}^1_R(P, X) = 0$ for all $X \in \mathscr{C}$. A left R-module F is called Gorenstein flat if $\operatorname{Tor}^1_R(X, F) = 0$ for any right R-module X which has finite dimensions; i.e., X belongs to the right counterpart of the class $\mathscr{C}$. For convenience, we use $\mathscr{G}\mathscr{F}$ to denote the class of all left Gorenstein flat R-modules.

It is appropriate to make a comment about Gorenstein projective modules here. Auslander introduced the modules of zero G-dimension over a Cohen−Macaulay local ring. Over such a ring, a finitely generated module P is called zero G-dimensional if there is an exact sequence

$$\cdots \to P^{-2} \to P^{-1} \to P^{0} \to P^{1} \to P^{2} \to \cdots$$

of projective modules with $A = \ker(P^0 \to P^1)$ such that $\operatorname{Hom}(-, P)$ leaves the complex exact for any projective module. We will see the consistency of the definitions of Gorenstein projective modules and modules of zero G-dimension over a Gorenstein ring in the next section.

For a Gorenstein injective G, it easy to see that $\operatorname{Ext}^i(X, G) = 0$ for all $i \geq 1$ and all $X \in \mathscr{X}$ by the properties of $\mathscr{C}$ and a shifting argument. For a left R-module M, a $\mathscr{G}\mathscr{F}$-cover (precover) is called a Gorenstein flat cover (precover).

PROPOSITION 1.4. *Let $\mathscr{X}$ be a class of modules. In addition, suppose $\mathscr{X}$ is closed under direct limits. Then for a left R-module M, the existence of an $\mathscr{X}$-precover of M implies the existence of an $\mathscr{X}$-cover.*

Proof. Although the statement of this result is quite different from that of Theorem 2.1 in [9], the whole process conducted there can be easily carried out here once we notice that the class $\mathscr{C}$ is closed under direct limits. More explicitly, if we replace injective modules by Gorenstein flat modules, and replace injective covers (precovers) by Gorenstein flat covers (precovers), we can state and prove three lemmas which correspond to those in [9] (Lemma 2.1, Lemma 2.2, and Lemma 2.3). Here we omit the details. ∎

Recall that every module can be essentially embedded into an injective module. For Gorenstein injective modules, we have the following fact.

PROPOSITION 1.5. *Let R be n-Gorenstein, M a left R-module. Then there is an exact sequence*

$$0 \to M \to G \to L \to 0$$

such that G is Gorenstein injective (i.e., $\in \mathscr{C}^{\perp}$) and L has finite dimensions (i.e., $L \in \mathscr{C}$).

Proof. The argument is essentially dual to the proof of Theorem 1.1 in (Auslander and Reiten [2]). For completeness, we give a proof here. First, taking a partial injective resolution of M, we have an exact sequence

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to G \to 0$$

with E_i $(0 \leq i \leq n-1)$ injective. We claim that G is Gorenstein injective. In fact, for any module $X \in \mathscr{C}$, by Proposition 1.1, $\operatorname{proj.dim}(X) \leq n$ since R is an n-Gorenstein ring. Then $\operatorname{Ext}^1(X, G) \cong \operatorname{Ext}^{n+1}(X, M) = 0$; hence, G is Gorenstein injective by the definition.

Now for an arbitrary Gorenstein injective module G (not necessary the same as above), we consider its injective cover E. It exists by Theorem 2.1 of [9] since R is Noetherian. Let $\varphi\colon E \to G$ be such an injective cover. We further claim that φ is a surjection and the kernel $K = \ker(E^{n-1} \to G)$ is also Gorenstein injective.

We first show that G must be a surjective image of an injective module. To see this, consider an exact sequence

$$0 \to Y \to P \to G \to 0$$

with P projective. Now we have an embedding $P \to E^*$ with E^* injective. Consider the following pushout diagram of $P \to G$ and $P \to E^*$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] & E^* \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & & D \arrow[r, equal] \arrow[d] & D \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

By Proposition 1.1, D has finite dimensions. But then the last column is split since G is Gorenstein injective and $\operatorname{Ext}^1(D, G) = 0$. So we have a surjection $X \to G$, and the composition of $E^* \to X$ and $X \to G$ gives us the desired surjection.

Therefore the injective cover of G, $\varphi\colon E \to G$ must be surjective by the condition (1) in the definition of injective covers (in Definition 1.1, the class $\mathscr{X}$ is replaced by the class of injective modules $\mathscr{E}$). Then there is an exact sequence $0 \to K \to E \to G \to 0$. By Proposition 1.3, $K \in \mathscr{X}^{\perp}$; i.e., $\operatorname{Ext}^1(E', K) = 0$ for any injective module E'. For any module $X \in \mathscr{C}$, by the exact sequence $0 \to K \to E \to G \to 0$, we have $\operatorname{Ext}^i(X, K) = 0$ for all $i \geq 2$ since $\operatorname{Ext}^i(X, G) \cong \operatorname{Ext}^{i+1}(X, K)$ and $\operatorname{Ext}^i(X, G) = 0$ for all $i \geq 1$. Now for every $X \in \mathscr{C}$, consider an embedding of X into an injective module E':

$$0 \to X \to E' \to X' \to 0$$

Clearly $X' \in \mathscr{C}$ since both X and E' are in $\mathscr{C}$. So by the above, $\mathrm{Ext}^2(X', K) = 0$. From this it follows that $\mathrm{Ext}^1(X, K) = 0$ since $\mathrm{Ext}^1(E', K)$ and $\mathrm{Ext}^2(X', K) = 0$. This means that K is Gorenstein injective.

Now repeating the same procedure for the kernel $K = \ker(E \to G)$, we get a surjective injective cover of K: $E(K) \to K$ with the kernel Gorenstein injective, and so on. Applying this to the exact sequence of M we had at the beginning, we get a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D \arrow[r] & E^0 \arrow[r] & E^1 \arrow[r] & \cdots \arrow[r] & E^{n-1} \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

Here E^{n-i} is an injective cover of $\ker E^{n-i-1} \to E^{n-i-2}$ with $E^{-2} = 0$, $E^{-1} = G$ for $i = 1, 2, \ldots, n$, and then D is also Gorenstein injective. The vertical maps exist since each E_{n-i} is injective and each E^{n-i} is an injective cover for $i = 1, 2, \ldots, n$.

We then have an associated exact sequence

$$0 \to M \to E_0 \oplus D \to E_1 \oplus E^0 \to \cdots \to E_{n-1} \oplus E^{n-2} \to G \oplus E^{n-1} \to G \to 0$$

with an exact subsequence $0 \to G \to G \to 0$. Therefore the quotient sequence

$$0 \to M \to E_0 \oplus D \to E_1 \oplus E^0 \to \cdots_{n-1} \oplus E^{n-2} \to E^{n-1} \to 0$$

is exact. Note that $E_0 \oplus D$ is Gorenstein injective. Set $G = E_0 \oplus D$, $L = G/M$. Then L is in the class $\mathscr{C}$. This provides us with the desired exact sequence $0 \to M \to G \to L \to 0$, and completes the proof. ∎

In Definition 1.2, if we specify the class $\mathscr{X}$ to be the class of Gorenstein injective modules ($\mathscr{C}^{\perp}$), we have the notion of Gorenstein injective envelopes (preenvelopes). Since all the injective modules are Gorenstein injective, it is easy to see that any Gorenstein injective preenvelope of an R-module M, $M \to G$, must be an injection. The above Proposition actually gives us a Gorenstein injective preenvelope of the given module M. In fact, let $0 \to M \to G \to L \to 0$ be the exact as in the Proposition. Then for $G' \in \mathscr{C}^{\perp}$,

$$\mathrm{Hom}_R(G, G') \to \mathrm{Hom}_R(M, G') \to \mathrm{Ext}^1_R(L, G') = 0$$

is exact and so $M \to G$ is a $\mathscr{C}^{\perp}$-preenvelope. With this in mind, we have:

DEFINITION 1.4. For a class $\mathscr{X}$ of left R-modules, an $\mathscr{X}$-preenvelope φ: $M \to X$ (with $X \in \mathscr{X}$) is called a special $\mathscr{X}$-preenvelope if $\mathrm{Coker}(\varphi) \in {}^{\perp}\mathscr{X}$ (note that for L above we have $L \in {}^{\perp}(\mathscr{C}^{\perp})$).

In a similar manner, we define special $\mathscr{X}$-precovers; i.e., an $\mathscr{X}$-precover $\varphi\colon X \to M$ is special if $\ker(\varphi) \in \mathscr{X}^{\perp}$.

Note that if $0 \to K \to X \to M \to 0$ is exact with $X \in \mathscr{X}$ and $K \in \mathscr{X}^{\perp}$, then $X \to M$ is an special $\mathscr{X}$-precover. Also, note that Wakamatsu's Lemma above says that every $\mathscr{X}$-cover is special.

2. GORENSTEIN PROJECTIVE MODULES

In this section, we will see that Gorenstein projective modules and Gorenstein injective modules behave like projective modules and injective modules.

THEOREM 2.1. *R is n-Gorenstein, A is a left R-module. Then the following are equivalent*:

(1) *A is Gorenstein projective; that is,* $\mathrm{Ext}^1_R(A, X) = 0$ *for any* $X \in \mathscr{C}$;

(2) $\mathrm{Ext}^i_R(A, X) = 0$ *for any* $i \geq 1$ *and* $X \in \mathscr{C}$;

(3) *For any left R-module M and any Gorenstein injective preenvelope of M, $M \to G$, any linear map $f\colon A \to L =$* $\mathrm{Coker}(M \to G)$ *can be lifted to $\bar{f}\colon A \to G$. That is, we have the following completed commutative diagram*:

$$\begin{tikzcd}
 & & & A \arrow[dl, dashed] \arrow[d] & \\
0 \arrow[r] & M \arrow[r] & G \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

(4) *There is an exact sequence*

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$$

of projective modules with $A = \ker(P^0 \to P^1)$ *such that* $\mathrm{Hom}_R(-, P)$ *leaves the complex exact for any projective module P*;

(5) *There is an exact sequence*

$$0 \to A \to P^0 \to \cdots \to P^{n-1}$$

with $P^0, P^1, \cdots, P^{n-1}$ projective.

Proof. (1) $\Rightarrow$ (2) This is easy by the property of $\mathscr{C}$ and the shifting argument. From this, we note that A is Gorenstein projective if and only if $\mathrm{Ext}^i_R(A, P) = 0$ for any $i \geq 1$ and any projective module P.

(3) $\Rightarrow$ (1) For any $X \in \mathscr{C}$, consider an extension of X by A

$$0 \to X \to H \to A \to 0$$

By Proposition 1.5, there is an exact sequence

$$0 \to X \to G \to L \to 0$$

such that G is Gorenstein injective (i.e., $\in \mathscr{C}^{\perp}$) and $L \in \mathscr{C}$. Easily, G also is in $\mathscr{C}$ because both X and L are in $\mathscr{C}$.

We claim that G is injective. Let $0 \to K \to E \to G \to 0$ be exact with $E \to G$ an injective cover. By the proof of Proposition 1.5, we see that the kernel K is also Gorenstein injective. Since G has finite dimensions, $\mathrm{Ext}^1(G, K) = 0$. So $0 \to K \to E \to G \to 0$ splits. Hence G is injective.

Therefore we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & G \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

Now, since $X \to G$ is a Gorenstein injective preenvelope of X, by the hypothesis, there is a linear map $\alpha\colon A \to G$ such that the following triangle is commutative

$$\begin{tikzcd}
 & A \arrow[dl, dashed] \arrow[d] \\
G \arrow[r] & L
\end{tikzcd}$$

Finally, note that H is the pullback of $A \to L$ and $G \to L$. Therefore, the first exact sequence is split. Hence $\mathrm{Ext}^1_R(A, X) = 0$ for any $X \in \mathscr{C}$. So A is Gorenstein projective.

(1) $\Rightarrow$ (3) We first show that for any left R-module M, if statement (3) holds for the case

$$0 \to M \to G \to L \to 0$$

with G Gorenstein injective and $L \in \mathscr{C}$, then statement (3) holds for general case; i.e., any linear map $A \to L$ can be lifted to a linear map $A \to G_1$ when G_1 is a Gorenstein injective preenvelope of M with the cokernel L_1.

In fact, suppose $0 \to M \to G_1 \to L_1 \to 0$ is an arbitrary Gorenstein injective preenvelope of M and $0 \to M \to G \to L \to 0$ is a special one with the cokernel L having finite dimensions (i.e., $L \in \mathscr{C}$). By the definition of Gorenstein injective preenvelope, we have the completed commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & G_1 \arrow[r, "p_1"] \arrow[d, "\alpha"] & L_1 \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & M \arrow[r] & G \arrow[r, "p"] & L \arrow[r] & 0
\end{tikzcd}$$

For any linear map $f\colon A \to L_1$, by the hypothesis, the linear map $\beta f\colon A \to L$ can be lifted to $g\colon A \to G$ such that $gp = \beta f$.

Note that G_1 is the pullback of $L_1 \to L$ and $G \to L$. Hence, there is a unique linear map $\bar{f}\colon A \to G_1$ such that $f = p_1\bar{f}$, $g = \alpha\bar{f}$. So statement (3) holds if it does for the special case.

Now we consider the special case, $0 \to M \to G \to L \to 0$ with G Gorenstein injective and L of finite dimensions. Let $P \to L$ be a surjective map with P projective. Then the kernel $K = \ker(P \to L)$ also has finite dimensions by Proposition 1.1. Obviously, $P \to L$ can be lifted to G. However, we have the following exact sequence

$$0 \to \mathrm{Hom}(A, K) \to \mathrm{Hom}(A, P) \to \mathrm{Hom}(A, L) \to \mathrm{Ext}^1(A, K) = 0$$

since A is Gorenstein projective and K has finite dimensions. It follows that any linear map $f\colon A \to L$ can be lifted to P and then to G. This finishes the implication (1) $\Rightarrow$ (2).

(1) $\Rightarrow$ (4) First of all, we need show that A can be embedded into a projective module.

Let E be an injective envelope of A. Let P be projective and a linear map $P \to E$ be surjective.

Consider the pullback diagram of $A \to E$ and $P \to E$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & X \arrow[r, equal] \arrow[d] & X \arrow[d] & & \\
0 \arrow[r] & D \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By this diagram and Proposition 1.1, it is easily seen that that X has finite dimensions. Since A is Gorenstein projective and so $\mathrm{Ext}_R^1(A, X) = 0$, the first vertical exact sequence is split. Therefore, A can be embedded into a projective module P.

Now since we have to produce a complex of projective modules for A as in statement (4), we first need to refine the projective embedding we have for A. Explicitly, we search for an embedding $A \to P$ with P projective such that any linear map $A \to P'$ with P' projective can be extended to P (or, we may call P a projective preenvelope of A). For if $A \to P$ is such an embedding and $0 \to A \to P \to G \to 0$ is exact, we will argue below that G is Gorenstein. This means that the whole process can be repeated.

To construct $A \to P$, we note R is right Noetherian (of course, right coherent). By Proposition 5.1 of [9], any left R-module has a flat preenvelope. In particular, A has a flat preenvelope $\varphi\colon A \to F$ with φ an injection since we have shown that A can be embedded into a projective module. For such a flat module F, we consider the exact sequence

$$0 \to H \to P \to F \to 0$$

where P is a projective module. Since both F and P have finite dimensions by Proposition 1.1, so does H. Now $H \in \mathscr{C}$ and $\mathrm{Ext}_R^1(A, H) = 0$ because A is Gorenstein projective. Then we have the exact sequence

$$0 \to \mathrm{Hom}_R(A, H) \to \mathrm{Hom}_R(A, P) \to \mathrm{Hom}_R(A, F) \to 0$$

In other words, $\varphi\colon A \to F$ can be lifted to $\overline{\varphi}\colon A \to P$. Clearly $\overline{\varphi}$ is also an injection.

Now let $0 \to A \to P \to G \to 0$ be the associated exact sequence. We claim that (a) any linear map $A \to P'$ with P' projective can be extended to P and (b) G is also Gorenstein projective. To see (a), for any linear map $\alpha\colon A \to P'$ with P' projective, consider the diagram

$$\begin{tikzcd}
 & & P \arrow[d] \\
0 \arrow[r] & A \arrow[ur, "\overline{\varphi}"] \arrow[r, "\varphi"] \arrow[dr, "\alpha"'] & F \arrow[d, dashed, "g"] \\
 & & P'
\end{tikzcd}$$

Note that P' is projective, so it is flat. Since $A \to F$ is a flat preenvelope, there is a linear map $g\colon F \to P'$ which makes the diagram commutative. Therefore, α can be extended to P.

To verify (b), consider the exact sequence

$$0 \to A \to P \to G \to 0$$

For any projective module P', $\mathrm{Hom}_R(-, P')$ leaves it exact by (a). It follows from this that $\mathrm{Ext}_R^1(G, P') = 0$ for any P' projective. On the other hand, by the exact sequence above, it is easy to see that for $i \geq 1$, $\mathrm{Ext}_R^{i+1}(G, P') = 0$. Consequently, $\mathrm{Ext}_R^i(G, P') = 0$ for any $i \geq 1$ and projective P', so G is also Gorenstein projective.

Now, by repeating the above procedure, we get the exact sequence

$$0 \to A \to P^0 \to P^1 \to \cdots$$

such that $\mathrm{Hom}_R(-, P)$ leaves the complex exact for any projective module P.

For the left side, consider an exact sequence

$$0 \to K \to P_0 \to A \to 0$$

with P_0 projective. Since $\mathrm{Ext}_R^1(A, P') = 0$ for any projective module P', it implies that $\mathrm{Hom}_R(-, P')$ leaves the sequence exact. Clearly, K is also Gorenstein projective. Then we can get the exact sequence in left by repeating the same procedure. Finally, pasting the left half and the right half together, we have established our desired exact sequence for any Gorenstein projective module A.

(4) $\Rightarrow$ (1) We can compute $\mathrm{Ext}_R^i(A, P')$ by the resolution of A provided by (4). Since it is left exact by $\mathrm{Hom}_R(-, P')$ for any projective module P', $\mathrm{Ext}_R^i(A, P') = 0$ for any $i \geq 1$.

(4) $\Rightarrow$ (5) It is trivial.

(5) $\Rightarrow$ (1) Consider the exact sequence

$$0 \to A \to P^0 \to P^1 \to \cdots \to P^{n-1} \to M \to 0$$

For any projective module P' and $i \geq 1$, $\mathrm{Ext}_R^i(A, P') \simeq \mathrm{Ext}_R^{i+n}(M, P') = 0$ since $\mathrm{inj.dim}_R P' \leq n$ by Proposition 1.1. Namely, A is Gorenstein projective. ∎

For the class of Gorenstein projective modules ${}^{\perp}\mathscr{C}$, we have the notion of Gorenstein projective precovers (covers). The following result shows that any module M over a n-Gorenstein ring has a special Gorenstein projective precover.

THEOREM 2.2. *Let R be is an n-Gorenstein ring, M a left R-module. Then there is an exact sequence*

$$0 \to L \to A \to M \to 0$$

such that A is Gorenstein projective and L has finite dimensions.

Proof. Given a left R-module M, let

$$0 \to A \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M \to 0$$

be a partial projective resolution of M. Then $\mathrm{Ext}_R^1(A, X) \cong \mathrm{Ext}_R^{n+1}(M, X) = 0$ for any $X \in \mathscr{C}$. This means that A is Gorenstein projective. Now, by (4) of Theorem 2.1, we have an exact sequence

$$0 \to A \to P^0 \to P^1 \to \cdots \to P^{n-1} \to D \to 0$$

such that $P^{i\prime}$s are projective and D is Gorenstein projective. Furthermore, for any projective module P, $\mathrm{Hom}_R(-, P)$ leaves the complex exact. Hence, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & P^0 & \longrightarrow & P^1 & \longrightarrow & \cdots & \longrightarrow & P^{n-1} & \longrightarrow & D & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Then, we can form the associated complex

$$0 \to A \to P^0 \oplus A \to P^1 \oplus P_{n-1} \to \cdots \to D \oplus P_0 \to M \to 0$$

It is exact by a spectral sequence argument. Note $0 \to A \xrightarrow{\mathrm{id}} A \to 0$ is a subcomplex which is exact. So the quotient complex

$$0 \to P^0 \to P^1 \oplus P_{n-1} \to \cdots \to P^{n-1} \oplus P_1 \to D \oplus P_0 \to M \to 0$$

is exact. This shows that $L = \ker(D \oplus P_0 \to M)$ has finite dimensions. Also $D \oplus P_0$ is Gorenstein projective. So we see that $D \oplus P_0 \to M$ is a special Gorenstein projective precover with the kernel of finite dimensions. ∎

We may ask whether every module has a Gorenstein projective cover. The answer is no in general. For example, let (R, m) be a regular local ring (i.e., $\mathrm{gl.dim}(R) < \infty$ or $K.\dim(R) = \dim[m/m^2 : R/m]$). Obviously, R is an n-Gorenstein ring and Gorenstein projective modules are just projective modules, and so Gorenstein projective covers are just projective covers. Although every finitely generated R-module has a projective (Gorenstein projective) cover, it is not true that every R-module has a Gorenstein projective cover because R is not perfect unless it is Artinian.

It is well known that a module P is projective if and only if the following diagram can be completed to a commutative diagram by a linear map $P \to E$

$$\begin{tikzcd}
 & & & P \arrow[dl, dashed] \arrow[d] & \\
0 \arrow[r] & M \arrow[r] & E \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

where $0 \to M \to E \to L \to 0$ is any exact sequence with E an injective module.

Statement (3) of Theorem 2.1 shows that a Gorenstein projective module acts similarly. Furthermore, for a Gorenstein injective module G, we have the dual statement.

THEOREM 2.3. *Let R be n-Gorenstein and R a left R-module. Then G is Gorenstein injective if and only if for any exact sequence $0 \to N \to A \to M \to 0$ with $A \to M$ a Gorenstein projective precover of M and any linear map $N \to G$, the diagram*

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & A \arrow[r] \arrow[dl, dashed] & M \arrow[r] & 0 \\
 & G & & &
\end{tikzcd}$$

can be completed to a commutative diagram by a linear map $A \to G$.

Proof. Suppose G is Gorenstein injective. By Theorem 2.2, there is a special Gorenstein projective precover of M

$$0 \to N_1 \to A_1 \to M \to 0$$

with A_1 Gorenstein projective (i.e., $A_1 \in {}^{\perp}\mathscr{C}$) and N_1 has finite dimensions (i.e., $N_1 \in \mathscr{C}$).

Let $0 \to N_1 \to E \to L \to 0$ be exact with E an injective envelope of N_1. Let $N_1 \to G$ be any linear map. Then there is a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & 0 \arrow[r] & N_1 \arrow[r] \arrow[dl] \arrow[d] & A_1 \arrow[r] \arrow[dl, dashed] & M \arrow[r] & 0 \\
G & & E \arrow[ll, dashed] \arrow[d] & & & \\
 & & L \arrow[d] & & & \\
 & & 0 & & &
\end{tikzcd}$$

The linear map $E \to G$ exists since $\mathrm{Ext}_R^1(L, G) = 0$. The linear map $N_1 \to E$ can be extended to $A_1 \to E$ since E is injective.

Now for any Gorenstein projective precover of M: $A \to M$, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_1 & \longrightarrow & A_1 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & A & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & G & & & & & &
\end{array}$$

Then, $N_1 \to N \to G$ can be extended to A_1. Also note that since A is the pushout of $N_1 \to N$ and $N_1 \to A_1$, there is a linear map h: $A \to G$ which extends the map $N \to G$.

Conversely suppose G satisfies the hypothesis. Let $X \in \mathscr{C}$ and let Y be an extension of G by X. We want to show that any such extension is split. Let $0 \to L \to P \to X \to 0$ be exact with P projective. Then since X and $P \in \mathscr{C}$, by Proposition 1.1 L is in $\mathscr{C}$. Hence it is easy to see that $P \to X$ is a special Gorenstein projective precover of X. Also note that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & G & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

However, by the hypothesis, $L \to G$ can be extended to $P \to G$. But this means that $0 \to G \to Y \to X \to 0$ is split. Since $0 \to G \to Y \to X \to 0$ was arbitrary, this says $\mathrm{Ext}_R^1(X, G) = 0$ for any $X \in \mathscr{C}$. Therefore, G is Gorenstein injective. ∎

We have noticed that an R-module M may not have a Gorenstein projective cover although it always has a Gorenstein projective precover. However, since every Gorenstein projective module is Gorenstein flat, we would like to shift our concern to Gorenstein flat modules. In the next section, we will use Theorem 2.2 to prove that every left module over a Gorenstein ring R has a Gorenstein flat precover and cover.

3. GORENSTEIN FLAT COVERS

We start with some lemmas.

LEMMA 3.1. *R is n-Gorenstein. The class of Gorenstein flat modules $\mathscr{GF}$ is closed under direct limits.*

Proof. This is obvious by the definition and the properties of the functors Tor. ∎

By this lemma and Proposition 1.4, in order to find a Gorenstein flat cover for a module M, we only need to find a Gorenstein flat precover of M.

LEMMA 3.2. *Let*

$$0 \to K \to F \to M \to 0$$
$$0 \to L \to G \to M \to 0$$

be exact where $f\colon F \to M$ and $k\colon G \to M$ are Gorenstein flat precovers of a module M. Then $G \oplus K \cong F \oplus L$.

Proof. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{g} & F & \xrightarrow{f} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle p} & & \| & & \\ 0 & \longrightarrow & L & \xrightarrow{h} & G & \xrightarrow{k} & M & \longrightarrow & 0 \end{array}$$

Here the linear map p is available such that $f = kp$ since $k\colon G \to M$ is a Gorenstein flat precover. This means that we have the above completed commutative diagram with exact rows.

Similarly, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{h} & G & \xrightarrow{k} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle q} & & \| & & \\ 0 & \longrightarrow & K & \xrightarrow{g} & F & \xrightarrow{f} & M & \longrightarrow & 0 \end{array}$$

We now can define two linear maps σ and ψ as follows:

$$\sigma : G \oplus K \to F \oplus L$$
$$(x', y) \mapsto \big(q(x') - g(y), -m(y) - h^{-1}(1 - pq)x'\big)$$
$$\psi : F \oplus L \to G \oplus K$$
$$(x, y') \mapsto \big(p(x) - h(y'), -l(y') - g^{-1}(1 - qp)x\big).$$

Note that $f = kp$ and $k = fq$, we have that $f = fqp$ and $k = kpq$, or equivalently,

$$f(1 - qp) = 0 \quad \text{and} \quad k(1 - pq) = 0.$$

Therefore, we have the following

$$(1 - qp)x \in \ker(f) = g(K), \qquad x \in F$$
$$(1 - pq)x' \in \ker(k) = h(L), \qquad x' \in G.$$

Therefore the maps above are well defined. It is easy to verify that $\sigma\psi = 1$ and $\psi\sigma = 1$ and then that

$$G \oplus K \cong F \oplus L. \quad \blacksquare$$

LEMMA 3.3. *Suppose* $0 \to A \to B \to C \to 0$ *is exact and that both* A *and* C *have Gorenstein flat covers. Also suppose* $A \in \mathscr{GF}^{\perp}$. *Then there is a commutative diagram with exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

such that $D = F \oplus G$ *and such that* $F \to A$ *and* $G \to C$ *are Gorenstein flat covers. Moreover, both* K *and* L *are in* $\mathscr{GF}^{\perp}$, *hence so is* N.

Proof. By Proposition 1.3, both K and L are in $\mathscr{GF}^{\perp}$. Since $G \to C$ can be lifted to $G \to B$, the usual construction and verification gives the diagram. $\blacksquare$

Before we state the next lemma, we recall the notion of pure injective modules. An exact sequence of left R-modules $0 \to S \to M \to N \to 0$ is called pure if $0 \to D \otimes S \to D \otimes M \to D \otimes N \to$ is exact for any right R-module D. A left R-module P is called pure injective if for any pure exact sequence $0 \to S \to M \to N \to 0$ and any linear map $f: S \to P$, f can be extended to $g: M \to P$; i.e., the following diagram can be completed to a commutative one:

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] \arrow[d, "f"'] & M \arrow[r] \arrow[dl, dashed] & N \arrow[r] & 0 \\ & P & & & \end{tikzcd}$$

Obviously all injective modules are pure injective. Pure injective modules were originally defined by Fuchs and Warfield for the study of algebraically compact modules (see [10, 21]). Then Jensen and Raynaud and Gruson used them in a nice relative homological theory (see [13, 16]). One of the most interesting facts about pure injective modules is that every module has a pure injective envelope (which agrees with the notion of envelope in Definition 1.2 when the class $\mathscr{X}$ is specified to be the class of all pure injective modules).

PROPOSITION 3.4. *Let $(-)^* = \mathrm{Hom}_Z(-, Q/Z)$ be the dual functor, here Q is the Abelian group of all rational numbers and Z is the group of all integers. Then an R-module M is pure injective if and only if it is a direct summand of the double dual M^{**}.*

Proof. See Warfield [21]. ∎

Now let go us back to Gorenstein flat covers of modules. We first want to show that every pure injective module has a Gorenstein flat cover.

LEMMA 3.5. *Let G be a Gorenstein injective module. Then $G^* = \mathrm{Hom}_Z(G, Q/Z)$ is Gorenstein flat.*

Proof. First note that for a left R-module M and a right R-module X, we have the canonical isomorphism

$$\mathrm{Ext}_R^1(X, \mathrm{Hom}_Z(M, Q/Z)) \cong \mathrm{Hom}_Z\big(\mathrm{Tor}_1^R(X, M), Q/Z\big).$$

Hence by the definition of Gorenstein injective modules and Gorenstein flat modules, an R-module D is Gorenstein flat if and only if D^* is Gorenstein injective.

In order to show that G^* is Gorenstein flat when G is Gorenstein injective, we only need show that G^{**} is Gorenstein injective. Since G is Gorenstein injective, it has an injective cover. Furthermore, by the proof of Proposition 1.5, there is an exact sequence

$$0 \to K' \to E \to G \to 0$$

such that $E \to G$ is an injective cover of G and such that K' is also Gorenstein injective. By repeatedly taking injective covers, we get an exact sequence

$$0 \to K \to E_0 \to E_1 \to \cdots \to E_{n-1} \to G \to 0$$

with $E_i (0 \le i \le n-1)$ injective and K Gorenstein injective. Then taking the double duals $(-)^{**}$ to this exact sequence, we have an exact sequence

$$0 \to K^{**} \to E_0^{**} \to E_1^{**} \to \cdots \to E_{n-1}^{**} \to G^{**} \to 0$$

Note that all E_i^{**} are injective (for it is not hard to show E_i^* is flat, and consequently E_i^{**} is injective) for $0 \le i \le n-1$. Now for an arbitrary module $X \in \mathscr{C}$,

$$\mathrm{Ext}^1(X, G^{**}) \cong \mathrm{Ext}^{n+1}(X, K^{**}) = 0$$

since $\mathrm{inj.dim}(X) \le n$ by Proposition 1.1. This shows that G^{**} is Gorenstein injective, and then G^* is Gorenstein flat by the remark above. ∎

LEMMA 3.6. *If R is n-Gorenstein and M is a pure injective left R-module, then we have an exact sequence*

$$0 \to K \to F \to M \to 0$$

such that both K and F are pure injective and $F \to M$ is a Gorenstein flat cover of M.

Proof. First of all, we show that any pure injective module M has a Gorenstein flat cover which is also pure injective. Since M is pure injective, it is a direct summand of M^{**}, where $(-)^* = \mathrm{Hom}_Z(-, Q/Z)$. Then we have a split exact sequence.

$$0 \to N \to M^{**} \to M \to 0$$

For the right R-module M^*, there is an exact sequence by Proposition 1.5.

$$0 \to M^* \to H \to L \to 0$$

such that H is Gorenstein injective and L has finite dimensions. Since $L^* = \mathrm{Hom}(L, Q/Z)$ and L has finite flat dimension, L^* has finite injective dimension. Hence by Proposition 1.1, L^* has finite dimensions. By Lemma 3.5, H^* is Gorenstein injective. Taking the duals, we get the following exact sequence:

$$0 \to L^* \to H^* \to M^{**} \to 0$$

Note that for any Gorenstein flat module F, $\mathrm{Ext}^1(F, L^*) \cong \mathrm{Hom}_Z(\mathrm{Tor}_1(L, F), Q/Z) = 0$ since $\mathrm{Tor}_1(L, F) = 0$. This means that L^* is in the orthogonal class $\mathcal{GF}^{\perp}$. From this it follows that H^* is a Gorenstein flat precover of M^{**}. Then, however, H^* also is a Gorenstein flat precover of M since M is a direct summand of M^{**}. Now, suppose $F \to M$ is a Gorenstein flat cover of M. It is easy to prove that F is a direct summand of H^*. Hence, F is pure injective.

We now claim that $K = \ker(F \to M)$ is also pure injective, let

$$0 \to L \to G \to N \to 0$$
$$0 \to K \to F \to M \to 0$$

be exact sequences where $G \to N$ and $F \to M$ are Gorenstein flat covers. It can be verified that $G \oplus F \to M^{**}$ is a Gorenstein flat precover of M^{**} with the kernel $X = L \oplus K$ since $M^{**} \cong N \oplus M$. Hence, X is in $\mathcal{GF}^{\perp}$ since both L and K are in this orthogonal class by Proposition 1.3. Now, we consider two Gorenstein flat precovers of M^{**}

$$0 \to X \to G \oplus F \to M^{**} \to 0$$
$$0 \to L^* \to H^* \to M^{**} \to 0$$

By Lemma 3.2, $X \oplus H^* \cong L^* \oplus G \oplus F$. Since both G and F are pure injective and $L^* = \mathrm{Hom}_Z(L, Q/Z)$ is also pure injective by a direct verification, it follows that X is pure injective. Therefore, K is pure injective. ∎

THEOREM 3.7. *Let R be n-Gorenstein and let M be a left R-module. If there is a resolution of length m by pure injective modules, i.e., an exact sequence*

$$0 \to M \to P_0 \to P_1 \to \cdots \to P_m \to 0$$

with each $P_i (0 \le i \le m)$ pure injective, then M has a Gorenstein flat cover. Furthermore, there exist exact sequences

$$0 \to K_0 \to F_0 \to M \to 0$$
$$0 \to K_1 \to F_1 \to K_0 \to 0$$
$$\cdots\cdots\cdots\cdots$$
$$0 \to K_{i+1} \to F_{i+1} \to K_i \to 0$$
$$\cdots$$

such that each F_i is Gorenstein flat and $K_i \in \mathcal{GF}^{\perp}$.

Proof. We only need to prove the last statement by induction on the length m. Suppose $m = 0$. Lemma 3.6 gives us the desired sequences. Now, suppose the conclusion is true for any left R-module having a resolution by pure injective modules of length k less than or equal to

$m-1$. Let M have such a resolution of length m. Consider the exact sequence

$$0 \to M \to P_0 \to N \to 0$$

where P_0 is pure injective and N has a resolution of length less than m. Then, by the induction hypothesis, we have the desired exact sequences for both P_0 and N as follows:

$$\begin{gathered}
0 \to L_0 \to F_0 \to N \to 0 \\
0 \to L_1 \to F_1 \to L_0 \to 0 \\
\cdots\cdots\cdots\cdot \\
0 \to L_{i+1} \to F_{i+1} \to L_i \to 0 \\
\cdots\cdots\cdots\cdot
\end{gathered}$$

and

$$\begin{gathered}
0 \to W_0 \to G_0 \to P_0 \to 0 \\
0 \to W_1 \to G_1 \to W_0 \to 0 \\
\cdots\cdots\cdots\cdot \\
0 \to W_{i+1} \to G_{i+1} \to W_i \to 0 \\
\cdots\cdots
\end{gathered}$$

Consider the pullback diagram of $P_0 \to N$ and $F_0 \to N$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \| & & \downarrow & & \\
 & & & & L_0 & = & L_0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & H & \longrightarrow & F_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

For the middle column, by using Lemma 3.2, we form the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \longrightarrow & K_0 & \longrightarrow & W_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & Z_0 & \longrightarrow & G_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_0 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Here $Z_0 = F_1 \oplus G_0$.

Then using the exact row $0 \to M \to H \to F_0 \to 0$ of the first diagram and the middle column of the preceding diagram, $0 \to K_0 \to Z_0 \to H \to 0$, we form a pullback of $Z_0 \to H$ and $M \to H$ and get the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K_0 & = & K_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & V_0 & \longrightarrow & Z_0 & \longrightarrow & F_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & H & \longrightarrow & F_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact rows and columns.

We know that K_0 is in $\mathscr{GF}^{\perp}$. From the middle row of the second diagram, we see that Z_0 is Gorenstein flat and from the middle row of the last diagram we see that V_0 is Gorenstein flat. This means that

$$0 \to K_0 \to V_0 \to M \to 0$$

is our first sequence for M. By using Lemma 3.2, we get the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_2 & \longrightarrow & K_1 & \longrightarrow & W_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_2 & \longrightarrow & V_1 & \longrightarrow & G_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_1 & \longrightarrow & K_0 & \longrightarrow & W_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Here, $V_1 = F_2 \oplus G_1$ is Gorenstein flat and K_1 is in $\mathscr{GF}^{\perp}$. Therefore, we have the second exact sequence

$$0 \to K_1 \to V_1 \to K_0 \to 0$$

By repeating the whole procedure, we then can construct all our desired exact sequences. ∎

In particular, the above result is applicable when M has finite injective dimension since injective modules are trivially pure injective. Of course, M has a Gorenstein flat cover by this theorem when M has finite pure injective dimension (see [16] for the definition of pure injective dimension). Before reaching our main theorem, we need one more lemma.

LEMMA 3.8. *Let R be n-Gorenstein and let M have finite injective dimension. Then there is an exact sequence*

$$0 \to M \to G \to L \to 0$$

such that G is in $\mathscr{GF}^{\perp}$ and L is Gorenstein flat.

Proof. Since any module M in $\mathscr{C}$ has finite injective dimension, by the preceding theorem it has a Gorenstein flat cover. Consider an exact sequence

$$0 \to M \to E \to N \to 0$$

where E is injective. Then N also has finite dimensions and so has a Gorenstein flat cover. Suppose $F \to N$ is a Gorenstein flat cover of N and

K is its kernel. Then we have the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & E \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We see that G is in $\mathscr{GF}^{\perp}$ since both K and E are in $\mathscr{GF}^{\perp}$. ∎

THEOREM 3.9. *Let R be n-Gorenstein. Then every left R-module has a Gorenstein flat cover.*

Proof. By Theorem 2.2, there is an exact sequence

$$0 \to L \to A \to M \to 0$$

such that L has finite dimensions and A is Gorenstein projective. As we mentioned before, A is Gorenstein flat. Now, by Lemma 3.8, we have an exact sequence

$$0 \to L \to G \to F \to 0$$

such that G is in $\mathscr{GF}^{\perp}$ and F is Gorenstein flat.

Consider the pushout diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & F \arrow[r, equal] \arrow[d] & F \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Now we have seen that D is Gorenstein flat since both F and A are Gorenstein flat, and that G is in $\mathscr{GF}^{\perp}$. This implies that $D \to M$ is a Gorenstein flat precover of M. Therefore, by Lemma 3.1 and Proposition 1.4, M has a Gorenstein flat cover. ∎

We now consider the question of preenvelopes and envelopes for the class $\mathscr{GF}^{\perp}$. We have the following result

THEOREM 3.10. *Let R be n-Gorenstein and M be a left R-module. Then M has a $\mathscr{GF}^{\perp}$-envelope.*

Proof. Let $0 \to M \to E \to N \to 0$ be exact with E injective. Let $F \to N$ be a Gorenstein flat cover of M. By an argument similar to the proof of Lemma 3.8, we have a special $\mathscr{GF}^{\perp}$-preenvelope of M

$$0 \to M \to G \to F \to 0$$

such that G is in $\mathscr{GF}^{\perp}$ and F is Gorenstein flat.

Since $\mathscr{GF}$ is closed under direct limits, there is an argument similar to that used to prove Proposition 1.4 that guarantees the existence of a $\mathscr{FG}^{\perp}$-envelope of M. ∎

COROLLARY 3.11. *R is n-Gorenstein. Then $^{\perp}(\mathscr{GF}^{\perp}) = \mathscr{GF}$.*

Proof. For any X in $^{\perp}(\mathscr{GF}^{\perp})$, by Theorem 3.9, X has a Gorenstein flat cover $F \to X$. Since the kernel is in $\mathscr{GF}^{\perp}$ and X is in $^{\perp}(\mathscr{GF}^{\perp})$, it is easy to see that

$$0 \to K \to F \to X \to 0$$

is split and then $X = F$ is Gorenstein flat. ∎

Remark 3.12. We can modify the procedure to show that over any n-Gorenstein ring R, every R-module has a flat cover.

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